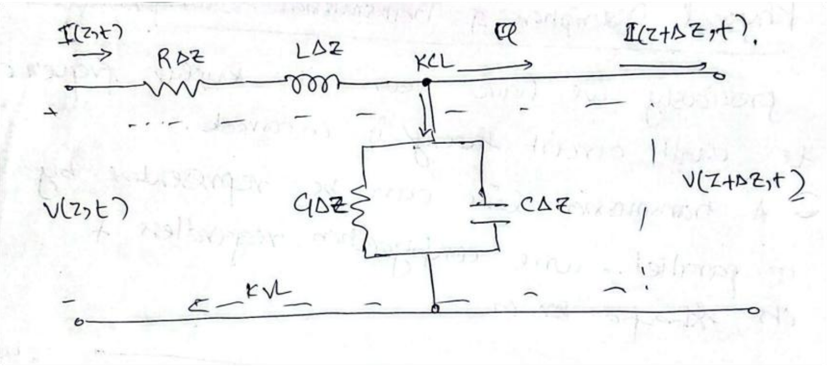
The Lumped Element Model

Formulating Equations for V(z,t) and I(z,t):



KVL

$$V(z,t) = R\Delta z\, I(z,t) + L\frac{\partial I(z,t)}{\partial t}\Delta z + V(z+\Delta z,t)$$

$$\frac{\Delta V}{\Delta z} = -\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right]$$

$$\lim_{\Delta z \to 0} \quad \frac{\partial V(z,t)}{\partial z} = -\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right]$$

KCL

$$I(z,t) = GV(z+\Delta z,t)\Delta z + C\frac{\partial V(z+\Delta z,t)}{\partial t}\Delta z + I(z+\Delta z,t)$$

$$\frac{\Delta I}{\Delta z} = -\left[GV(z+\Delta z,t) + C\frac{\partial V(z+\Delta z,t)}{\partial t}\right]$$

$$\lim_{\Delta z \to 0} \quad \frac{\partial I(z,t)}{\partial z} = -\left[GV(z,t) + C\frac{\partial V(z,t)}{\partial t}\right]$$

$$\frac{\partial^2 V(z,t)}{\partial z^2} = (LC)\,\frac{\partial^2 V(z,t)}{\partial t^2} + (LG+RC)\,\frac{\partial V(z,t)}{\partial t} + RG\,V(z,t) \qquad (3)$$

$$\frac{\partial^2 I(z,t)}{\partial z^2} = (LC)\,\frac{\partial^2 I(z,t)}{\partial t^2} + (LG+RC)\,\frac{\partial I(z,t)}{\partial t} + RG\,I(z,t) \qquad (4)$$

Lossless Propagation

Verification With Wave Equation:

Substitute (6) into the wave equation:

$$\frac{\partial^2 V}{\partial z^2} = LC\frac{\partial^2 V}{\partial t^2}.$$

Compute the derivatives:

$$\frac{\partial V}{\partial z} = -\tfrac{1}{v}f_1' + \tfrac{1}{v}f_2', \quad \frac{\partial^2 V}{\partial z^2} = \tfrac{1}{v^2}(f_1'' + f_2''),$$
$$\frac{\partial V}{\partial t} = f_1' + f_2', \quad \frac{\partial^2 V}{\partial t^2} = f_1'' + f_2''.$$

Thus,

$$\frac{1}{v^2}(f_1'' + f_2'') = LC\,(f_1'' + f_2'').$$

This condition is satisfied if

$$\frac{1}{v^2} = LC \quad \Longrightarrow \quad v = \frac{1}{\sqrt{LC}}. \qquad (7)$$

Current Solution

From the first-order telegrapher's equation

$$\frac{\partial V}{\partial z} = -L\frac{\partial I}{\partial t},$$

substitute (6):

$$\frac{\partial V}{\partial z} = -\frac{1}{v}f_1' + \frac{1}{v}f_2' = -L\frac{\partial I}{\partial t}.$$

Hence,

$$\frac{\partial I}{\partial t} = \frac{1}{Lv}\left(f_1' - f_2'\right).$$

Integrating with respect to t gives

$$I(z,t) = \frac{1}{Lv}\left[f_1\!\left(t - \tfrac{z}{v}\right) - f_2\!\left(t + \tfrac{z}{v}\right)\right] + C, \qquad (8)$$